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NUMERICAL APPROXIMATIONS OF GENERALIZED SOLUTIONS OF THE HAMILTON–JACOBI EQUATIONS†

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The Cauchy problem for a first-order partial differential equation whose left-hand side is a homogeneous function of the vector of derivatives, with the time derivative occurring additively, is considered. The boundary conditions are specified at the right end of the time interval. The solution of a differential game over a fixed time interval with a terminal functional is reducible to a problem of this type. The traditional difference method for constructing the solution of a boundary-value problem is not applicable, because the generalized solution need not be smooth. A mathematical technique, based on methods of solving game problems, is proposed. The resultant computational scheme, whose validity is established in three theorems, is based on a rectangular space mesh and a subdivision of the time interval. Unlike the classical approach, the scheme uses not finite differences but subdifferentials of the convex hulls of functions approximating the value function. Copyright © 1996 Elsevier Science Ltd.

There are several definitions of generalized solutions of the Hamilton–Jacobi equations [1–4] which, though different in form, are essentially equivalent. These definitions are based on replacing the equation by a pair of (differential) inequalities. This paper is based on constructions from the theory of positional differential games [5, 6], which was developed by N. N. Krasovskii and his coworkers and combines methods for solving a broad range of problems—from existence theorems to the design of numerical algorithms. It should be noted that research in recent years, carried out using different approaches (see, for example, [7–10]), was preceded by work of various authors in the 1950s and 1960s. This paper continues the studies in [1–6, 11–18].[1]

1. FORMULATION OF THE PROBLEM

Consider the Cauchy problem

$$\frac{\partial w}{\partial t}(t,x)+h(t,x,\nabla w(t,x))=0 \quad \left(\nabla w(t,x)=\left(\frac{\partial w}{\partial x_1}(t,x),\ldots,\frac{\partial w}{\partial x_n}(t,x)\right)\right)$$
$$w(\vartheta,x)=\sigma(x),\quad t\in[0,\vartheta),\quad x\in R^n \tag{1.1}$$

where $h(t, x, s)$ is the Hamiltonian.

The problem of approximating a generalized (min-max, viscosity) solution of problem (1.1) will be studied in the context of the differential game (DG)

$$\dot{x}=f(t,x,u,v)\equiv f^1(t,x,u)+f^2(t,x,v) \tag{1.2}$$
$$t\in[0,\vartheta],\quad u\in P\subset R^p,\quad v\in Q\subset R^q$$

where x is the n-dimensional phase vector of the system, u and v are the vectors of controls of the first and second players, respectively, and P and Q are compact sets. The Hamiltonian of the dynamical system is

$$h(t,x,s)=\min_{u\in P}\langle s,f^1(t,x,u)\rangle+\max_{v\in Q}\langle s,f^2(t,x,v)\rangle$$

where $\langle s,f\rangle$ is the inner product of the vectors s and f.

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The right-hand side of system (1.2) satisfies the conditions for the existence, uniqueness and extendibility of a solution. As the performance index of the DG we consider the terminal functional

$$\gamma(x(\cdot)) = \sigma(x(\vartheta)) \tag{1.3}$$

where $\sigma(\cdot): R^n \Rightarrow R$ is a function satisfying a local Lipschitz condition. The functional γ associates with the actually realized motion $x(\cdot) = \{x(t), 0 \leq t \leq \vartheta\}$ of system (1.2) a number $\sigma(x(\vartheta))$—the payoff of the DG.

With the above conditions imposed on the right-hand side of system (1.2), a value function $(t, x) \Rightarrow w(t, x)$: $[0, \vartheta] \times R^n \Rightarrow R$ of the DG (1.2), (1.3) exists, which is a generalized solution of problem (1.1). The value function is the only function which, given the payoff (1.3), simultaneously satisfies the u- and v-stability properties. The property of u-stability (v-stability) means that the epigraph (hypograph) of the value function is weakly invariant under a certain family of differential inclusions—the family of characteristic inclusions for (1.1). Weakly invariant sets may be constructed using a stable absorption operator (SAO) [14]. It should be noted here that there is some arbitrariness in the choice of the family of characteristic inclusions, which will be used later in constructing the SAO.

The technique presented in this paper assumes the construction of the restriction of the function w to a bounded set $D \subset [0, \vartheta] \times R^n$. This set will be chosen as a stable bridge in the conflict problem of approach described by Eq. (1.2), with a set

$$\hat{M} = O(\hat{x}, \hat{r}) = \{x \in R^n: \ \| x - \hat{x} \| \leq \hat{r}\}$$

defined as a sphere of radius $\hat{r}$ with centre at a certain arbitrary point $\hat{x}$, where $\hat{r}$ is a sufficiently large positive number.

Definition 1. A multivalued mapping

$$t \Rightarrow D(t) \subset R^n, \qquad t \in [0, \vartheta]$$

is called a stable bridge in problem (1.2), (1.3) if $D(\vartheta) \subset \hat{M}$, while its graph $\{(t, x) \in R \times R^n: t \in [0, \vartheta], x \in D(t)\}$ is closed and weakly invariant under the differential inclusions
$\dot{x} \in F(t, x, s)$ for any $s \in S_n$

$$F(t, x, s) = \{f \in F: \langle s, f \rangle \geq h(t, x, s)\}, \quad F = \{f \in R^n: \| f \| \leq K\}$$

where $S_n = \{s \in R^n: \| s \| = 1\}$ is the unit sphere and the constant K is chosen so that

$$K > 2 \sup_{(t, x, u, v) \in \hat{D} \times P \times Q} \| f(t, x, u, v) \| \tag{1.4}$$

with $\hat{D}$ a bounded set in $[0, \vartheta] \times R^n$, containing D.

Note that under the above restrictions on the right-hand side of Eq. (1.2), the family of multivalued mappings generating D satisfies the following conditions

$$\{(t, x) \Rightarrow F(t, x, s): s \in S_n\} \tag{1.5}$$

$\hat{A}1$. For any $(t, x, s) \in \hat{D} \times S_n$, the set $F(t, x, s)$ is convex, closed and satisfies the embedding relation $F(t, x, s) \subset F$.

$\hat{A}2$. For any $(t, x, s) \in \hat{D} \times S_n$

$$\max_{q \in S_n} \min_{f \in F(t, x, q)} \langle s, f \rangle = h(t, x, s).$$

$\hat{A}3$. Given the mapping $(t, x, s) \Rightarrow F(t, x, s)$, a function $\delta \Rightarrow \hat{\omega}(\delta): R \Rightarrow R(\hat{\omega}(\delta) \to 0$ exists (as $\delta \to 0$) such that

$$\operatorname{dist}(F(t_1, x, s), F(t_2, y, s)) \leq \hat{\omega}(| t_1 - t_2 | + \| x - y \|)$$

for all (t_1, x) and (t_2, y) in $\hat{D}$ and any s in S_n, where $\operatorname{dist}(F_1, F_2)$ is the Hausdorff distance between sets F_1 and F_2.

$\hat{A}4$. A number $\lambda_F \in [0, +\infty)$, exists such that, for any (t, x) and $(t, y) \in \hat{D}$ and any $s \in S_n$

$$\operatorname{dist}(F(t, x, s), F(t, y, s)) \leq \lambda_F \| x - y \|$$

One more fact needs to be mentioned. A continuous function $w(\cdot)$ is a value function for the DG (1.2), (1.3) if and only if the set $W = \operatorname{epi} w$ ($W = \operatorname{hypo} w$) is a stable bridge in the problem of approach with target set $M = \operatorname{epi} \sigma$ ($M = \operatorname{hypo} \sigma$) being solved by the player implementing the control u (the control v), for the extended system

$$\dot{x} = f(t, x, u, v), \quad \dot{\chi} = 0 \tag{1.6}$$

Here epi σ and hypo σ are the epigraph and hypograph, respectively, of the function σ, $\chi \in R$.

It is obvious that this statement remains true when one considers the restriction of w to a stable bridge D

$$w(\cdot): D \Rightarrow R$$

In what follows, we shall refer to the problem of constructing the epigraph of the restriction to D of w as Problem 1, and to that of constructing the hypograph of the restriction of w as Problem 2. As the solutions of these two problems are approached in similar ways, we shall only describe the construction for one of them, say Problem 1.

2. FAMILY OF FORMS OF A STABLE ABSORPTION OPERATOR FOR THE EXTENDED SYSTEM

In this section we shall propose correct forms of SAOs for constructing the set W—the epigraph of the restriction to D of the value function of the DG (1.2), (1.3); by "correct" here we mean "compatible with the theorems of the theory of DGs". We will consider a collection of families of multivalued mappings and investigate the properties of the families. It will be shown that each representative of the collection induces a SAO. This will enable us to pick out a set of forms of SAOs, each of which solves Problem 1.

We introduce the following notation

$$z = (x, \chi), \quad x \in R^n, \quad \chi \in R$$

$$\bar{f}(t, z, u, v) = (f(t, x, u, v), 0), \quad f(t, x, u, v) \in R^n, \quad 0 \in R$$

If $\Omega \subset R^n \times R$, then pr Ω denotes the orthogonal projection of Ω onto R^n.

Let us consider a DG for the extended dynamical system (1.6)

$$\dot{z} = \bar{f}(t, z, u, v), \quad z \in R^{n+1}, \quad t \in [0, \vartheta], \quad u \in P, \quad v \in Q \tag{2.1}$$

We will take as the objective set

$$M = \operatorname{epi} \sigma_{D(\vartheta)} \tag{2.2}$$

i.e. the epigraph of the restriction of the payoff function to the set $D(\vartheta)$.

A stable bridge W for the approach problem (1.2), (1.3) will accomplish the solution of Problem 1.

The Hamiltonian of system (2.1) is given by

$$H(t, z, l) = \min_{u \in P} \max_{v \in Q} \langle l, \hat{f}(t, z, u, v) \rangle, \quad l \in R^{n+1}$$

The symbol S_{n+1} will denote the set of vectors $\{l \in R^{n+1}: \| l \| 1\}$.

Construct the direct product of the sets F and the interval $[-c, c]$

$$F^c = F \times [-c, c], \quad c \in [0, +\infty)$$

Define in (t, z)-space a domain D^* that contains W a priori: $D^* = \hat{D} \times R$.

Let us consider a collection of families of multivalued mappings, parametrized by the values of $c \in [0, +\infty)$

$$\begin{gathered}\{(t,z) \Rightarrow F^c(t,z,l): l \in S_{n+1}\} \\ (t,z) \in D^*, \quad F^c(t,z,l) = \{\bar{f} \in F^c: \langle l, \bar{f} \rangle \geqslant H(t,z,l)\}\end{gathered} \tag{2.3}$$

If the family of mappings (1.5) satisfies conditions Â1–Â4, then any family of mappings in the collection (2.3) will satisfy analogous conditions, namely, the following conditions A1–A4.

A1. For any $c \geqslant 0$ and any $(t, z, l) \in D^* \times S_{n+1}$, the set $F^c(t, z, l)$ is convex, closed and satisfies the embedding relation $F(t, z, l) \subset F^c$.

A2. For any $c \geqslant 0$ and any $(t, z, l) \in D^* \times S_{n+1}$

$$\max_{q \in S_{n+1}} \min_{\bar{f} \in F^c(t,z,q)} \langle l, \bar{f} \rangle = H(t,z,l) = h(t, \operatorname{pr} z, \operatorname{pr} l)$$

A3. For any $c \geqslant 0$, given the mapping $(t, z, l) \Rightarrow F^c(t, z, l)$, a constant $\nu(c)$ exists such that

$$\operatorname{dist}(F^c(t_1,z,l), F^c(t_2,z',l)) \leqslant \nu(c)\hat{\omega}(|t_1 - t_2| + \|\operatorname{pr} z - \operatorname{pr} z'\|)$$

for all $(t_1 z)$ and $(t_2 z')$ in D^* and any l in S_{n+1}.

A4. For any $c \geqslant 0$, a constant $\lambda = \lambda(c) \in (0, +\infty)$ exists such that, for any (t, z) and $(t, z') \in D^*$, and any $l \in S_{n+1}$

$$\operatorname{dist}(F^c(t,z,l), F^c(t,z',l)) \leqslant \lambda \|\operatorname{pr} z - \operatorname{pr} z'\|$$

We know [17] that any family of mappings satisfying conditions A1–A4 induces a SAO for the corresponding approach problem. In the present case, all such families of mappings are treated as a single collection, each being singled out by the value of the non-negative parameter c.

Definition 2. A stable absorption operator

$$\pi^c(t_*, t^*, \cdot)\,(0 \leqslant t_* < t^* \leqslant \vartheta)$$

for problem (2.1), (2.2) is a mapping

$$B \Rightarrow \pi^c(t_*, t^*, B): 2^{R^{n+1}} \Rightarrow 2^{R^{n+1}}$$

given by the relationship

$$\pi^c(t_*, t^*, B) = \{z_* \in R^{n+1}: B \cap Z^c(t^*; t_*, z_*, l) \neq \emptyset$$

for all $l \in S_{n+1}$, where $Z^c(t^*; t_*, z_*, l)$ is the set of all points in R^{n+1} for which, at time t, one obtains solutions $z(t)$ $(t_* \leqslant t \leqslant t^*, z(t_*) = z_*)$ of the differential inclusion

$$\dot{z} \in F^c(t,z,l), \quad l \in S_{n+1}$$

Definition 3. A set $W \subset D^*$ is called a stable bridge in Problem 1 of approach to a closed target $M \subset R^{n+1}$ is the following conditions hold

1. $W(\vartheta) \subset D$.
2. $W(t_*) \subset \pi^c(t_*, t^*, W(t^*))$ for all $t_*, t^*(0 \leqslant t_* < t^* \leqslant \vartheta)$.

Thus, we have described a family $B \Rightarrow \tilde{\pi}^c(t_*, t^*, B)$: $2^{R_{n+1}} \Rightarrow 2^{R_{n+1}}$ of forms of SAOs, each of which may be included in order to construct epi w_D.

3. APPROXIMATIONS OF A STABLE ABSORPTION OPERATOR FOR THE EXTENDED SYSTEM

Following the earlier approach [12], we shall define the notion of an approximating form of a stable absorption operator for Problem 1.

Definition 4. An approximating form of a stable absorption operator $\tilde{\pi}^c(t_*, t^*, \cdot)$ ($c \geq 0$; $0 \leq t_* < t^* \leq \vartheta$) for Problem 1 is a mapping $B \Rightarrow \tilde{\pi}^c(t_*, t^*, B)$: $2^{R_{n+1}} \Rightarrow 2^{R_{n+1}}$ given by

$$\tilde{\pi}^c(t_*, t^*, B) = \{z_* \in R^{n+1} : B \cap \tilde{Z}_1^c(t^*; t_*, z_*) \neq \emptyset$$

for all $l \in S_{n+1}\}$, where $\tilde{Z}_1^c(t^*; t_*, z_*) = z_* + (t^* - t_*)F^c(t_*, z_*, l)$.

Let $\Gamma = \{0, t_1, \dots, t_N = \vartheta\}$ be a partition of the interval $[0, \vartheta]$.

We will now define a system of sets that approximates a maximum stable bridge W in Problem 1.

Definition 5. An approximating system of sets in Problem 1 is a collection of sets $\{\tilde{W}^c(t_i) \subset R^{n+1}: t_i \in \Gamma\}$ such that

$$\tilde{W}^c(t_N) = M_{\varepsilon(N,c)}$$

$$\tilde{W}^c(t_i) = \tilde{\pi}^c(t_i, t_{i+1}, \ \tilde{W}^c(t_{i+1})), \quad i = N-1, \dots, 0$$

where $c \geq 0$ and B_ε is a closed ε-neighbourhood of the set B

$$B_\varepsilon = \{b \in R^{n+1}: \ \min_{a \in B} \rho(a, b) \leq \varepsilon\}$$

$$\rho(a, b) = \max\{|a_i - b_i|: \quad i = 1, \dots, n+1\}$$

The number $\varepsilon(N, c)$ is found from the recurrent relations

$$\varepsilon(i+1, c) = \Delta_i \nu(c) \hat{\omega}(\Delta_i(1+K)) + (1 + \lambda \Delta_i)\varepsilon(i, c)$$

$$\Delta_i = t_{i+1} - t_i, \quad \varepsilon(0, c) = 0$$

where K is the constant in condition (1.4), $\hat{\omega}(\cdot)$ is the function from condition $\hat{A}3$, $\lambda = \lambda(c)$ is the constant from condition A4, and $\nu(c)$ is the constant from condition A3.

We now define the limit of an approximating system of sets.

Consider a sequence $\{\Gamma_j: j = 1, 2, \dots\}$ of partitions of the interval $[0, \vartheta]$ whose diameters $\Delta_j = \max_i |t_{i+1} - t_i|$ ($i = 1, \dots, N(j) - 1$) tend to zero as $j \to +\infty$.

Definition 6. Let $\tilde{W}^c$ denote the set of points $(t_*, z_*) \in D^*$ for which a sequence

$$\{(t_j, z_j): \ t_j = t_j(t_*) \in [0, \vartheta], \quad z_j \in \tilde{W}^c(t_j), \quad \lim_{j \to +\infty} z_j = z_*\}$$

exists, where

$$t_j(t_*) = \begin{cases} \min\limits_{(t_i \in \Gamma_j, \ t_i > t_*)} t_i, & t_* < \vartheta \\ t_*, & t_* = \vartheta \end{cases}$$

The set $\tilde{W}^c$ will be called the limit of the approximating system $\{\tilde{W}_j^c(t_i): t_i \in \Gamma_j\}$ as $j \to +\infty$.

Theorem 1. If the mapping $(t, z, l) \Rightarrow F^c(t, z, l)$ ($c \geq 0$) ($(t, z, l) \in D^* \times S_{n+1}$) satisfies conditions A1–A4, the set $\tilde{W}^c$ is the maximum stable bridge W for Problem 1.

The proof of Theorem 1 is similar to the proof of the convergence of the constructions of [12].

4. STEP OPERATOR

We will now enumerate some properties of approximating forms of SAOs over a single step of the partition of the time interval. Before we describe them, a few facts should be pointed out. The value function of the DG is constructed using difference procedures. In our case difference procedures require that we know how to find such increments as $\delta w(t, x) = w(t + \Delta, x) - w(t, x)$, where $t \in [0, \vartheta)$, $\Delta > 0$, $t + \Delta \in [0, \vartheta]$, $x \in D(t) \cap D(t + \Delta)$, to a satisfactory accuracy.

Note that not every form π^c ($c \in [0, +\infty)$) of a SAO is suitable for that purpose. Thus, a form π^c of a SAO with $c = 0$ "propagates" its action along the solution levels. This means that, using π^0, a representation of the value function is formed by constructing its level sets, for which one has to solve equations for y of the form $w(t + \Delta, x) - w(t, y) = 0$, where $w(t + \Delta, x) = \text{const}$. Below, in particular, it will be shown that for sufficiently large values of the parameter c, estimating the rate of change of the value function, the corresponding approximating forms of the SAO enable one approximately to compute increments of the form $\delta w(t,x)$. In such situations the forms of the operator act in an equivalent manner and permit the use of locally convex hulls.

We now define a conical set in (t, x)-space

$$\bar{D} = \{(t,x) \in [0,\vartheta] \times R^n: \ t \in [t_0,\vartheta], \ x \in (\hat{x} + (t - t_0)F\}$$

$$t_0 = \max\{0, \vartheta - \hat{r} / K\}.$$

The constructions imply the following properties

1. $\bar{D}$ is strongly invariant with respect to the inclusion $\dot{x} \in F$; consequently, $\bar{D} \subset D$.

2. The section $\bar{D}(t)$ of $\bar{D}$ at each time t is a sphere with centre at $\hat{x}$ and radius $r = r(t) = \hat{r} - (\vartheta - t)K$.

The constructions described below, which are involved in the study of properties of the family of forms $\{\pi^c(\cdot), c \geqslant 0\}$ of a SAO, are considered "above" the set $\bar{D}$.

Definition 7. Let $t \in [t_0, \vartheta)$, let $\Delta > 0$ be a number such that $(t + \Delta) \in [t_0, \vartheta]$, and let $c \geqslant 0$. The step operator $\pi^c_\Delta(\cdot)$ is the mapping $2^{R_{n+1}} \Rightarrow 2^{R_{n+1}}$ defined by

$$\pi^c_\Delta(B) = \{z_* \in R^{n+1}: \ B \cap \tilde{Z}^c(t + \Delta; \ t, z_*, l) \neq 0 \quad \forall l \in S_{n+1}\}$$

where $B \subset R^{n+1}$, $\tilde{Z}^c(t + \Delta; t, z_*, l) = z_* + \Delta F^c(t, z_*, l)$.

We will now characterize the properties of the step operates π^c_Δ. Throughout, the choice of the numbers t and Δ is governed by the condition $t + \Delta \leqslant \vartheta$.

We shall say that a set $\Omega \subset R^{n+1}$ is upper stable [19] if the inclusion $(x, \mu_*) \in \Omega$ implies the inclusion $(x, \mu_*) \in \Omega$, where $\mu^* > \mu_*$.

Property 1. Assume that $c \in [0, +\infty)$, the function φ is defined and bounded in the set $\bar{D}(t + \Delta)$. Then the set $\pi^c_\Delta(\text{epi } \varphi)$ is non-empty and upper stable.

Upper stable sets generate functions that are defined by applying the operation inf to an appropriate set of numbers

$$\psi^c_\Delta(x) = \inf\{\chi: (x, \chi) \in \pi^c_\Delta(\text{epi } \varphi), \ \ x \in \bar{D}(t)\}, \quad c \geqslant 0$$

Thus, the operators π^c_Δ map epigraphs of functions into sets that may be treated as epigraphs of functions.

Property 2. Assume that $c \geqslant 0$ and that the function φ is defined and continuous in the set $\bar{D}(t + \Delta)$. Then, for any point $x \in \bar{D}(t)$, a point $y \in O(x, K\Delta)$ exists such that

$$\varphi(y) = \psi^c_\Delta(x).$$

Property 3. Assume that $c \geqslant 0$ and that the function φ: $\bar{D}(t + \Delta) \Rightarrow R$ satisfies a Lipschitz condition with constant $\lambda = \lambda(\bar{D}(t + \Delta))$. Then, for any point $x \in \bar{D}(t)$

$$| \psi^c_\Delta(x) - \varphi(y) | \leqslant 2\lambda K\Delta \quad \forall y \in O(x, K\Delta)$$

We now formulate conditions under which the step operators in the set $\{\pi^c_\Delta, c \geqslant 0\}$ act in an equivalent manner, in the sense that they generate identically equal functions.

We recall the definition of the convex hull of a function [20].

Definition 8. Let φ be a function defined in the set $\bar{D}(t + \Delta)$ in R^n. Consider the restriction $\varphi_{O(x, K\Delta)}$ of φ to the sphere $O(x, K\Delta), x \in \bar{D}(t)$. Then co φ will denote the locally convex hull (LCH) of the function $\varphi_{O(x, K\Delta)}$

$$\text{co}\,\varphi(y) = \inf \left\{ \sum_{i=1}^{n+1} \alpha_i \varphi(y^{(i)}): \ \alpha_i \geqslant 0, \ \ i = 1, \ldots, n+1; \right.$$

$$\sum_{i=1}^{n+1} \alpha_i = 1, \quad \sum_{i=1}^{n+1} \alpha_i y^{(i)} = y, \quad y^{(i)} \in O(x, K\Delta)\Bigg\}$$

We emphasize that the LCH of a function φ depends on two parameters: a point $x \in \bar{D}(t)$ and a number $K\Delta$—the radius of a sphere with centre at x. For brevity we have not identified these parameters in the notation for LCHs.

The theorem formulated next uses known corollaries [12] of conditions A1–A4, properties 1–3, the definition of LCH and separation theorems from convex analysis. We first define the function $\bar{\psi}^c_\Delta(\cdot)$: $\bar{D}(t) \Rightarrow R$, $c \geqslant 0$

$$\bar{\psi}^c_\Delta(x) = \inf\{\chi\colon (x, \chi) \in \pi^c_\Delta(\text{epi co}\,\varphi_{O(x, K\Delta)})\}$$

Theorem 2. If φ: $\bar{D}(t + \Delta) \Rightarrow R$ is a function satisfying a Lipschitz condition with constant λ, $x \in \bar{D}(t)$, then, for any $c \geqslant 2\lambda K$ and any sufficiently small number $\Delta > 0$

$$\pi^c_\Delta(\text{epi}\,\varphi_{O(x, K\Delta)}) = \pi^c_\Delta(\text{epi co}\,\varphi) = \pi^0_\Delta(\text{epi co}\,\varphi)$$

The theorem states, in particular, that for any $c \geqslant 2\lambda K$

$$\psi^c_\Delta(x) = \bar{\psi}^c_\Delta(x) = \bar{\psi}^0_\Delta(x), \quad x \in \bar{D}(t)$$

5. FORMULAE OF THE DIFFERENCE CALCULUS

In this section we will translate the preceding description of functions constructed using step operators on sections of a strongly invariant set $\bar{D}$ from set-theoretic language into the language of analytic formulae. Thus, with the assumptions of Theorem 2, the functions $\bar{\psi}^c_\Delta(\cdot)$: $\bar{D}(t) \Rightarrow R$ are identical. One them has the representation

$$\psi^c_\Delta(x) = \max_{s \in S_n} \min_{f \in F(t,x,s)} \text{co}\,\varphi(x + \Delta f), \quad x \in \bar{D}(t), \quad c \geqslant 2\lambda K \tag{5.1}$$

The proof of this equality relies on the definition of step operators, on the families of mappings (1.5) and (2.3), and on condition A2.

We will need some properties of the max-min of the LCH of a function φ. For this purpose we define sets

$$F_{\text{ext}}(t, x) = \left\{ f_{\text{ext}} \in F\colon \max_{s \in S_n} \min_{f \in F(t,x,s)} \text{co}\,\varphi(x + \Delta f) = \text{co}\,\varphi(x + \Delta f_{\text{ext}}) \right\}, \quad (t, x) \in \bar{D}$$

Let K_0 be a constant such that

$$\max_{(t,u,v) \in [t_0, \vartheta] \times P \times Q} \| f(t, x, u, v) \| \leqslant K_0 < K/2$$

Property 4

$$F_{\text{ext}}(t, x) \cap O(0, K_0) \neq 0, \quad \forall (t, x) \in \bar{D} \tag{5.2}$$

This relation indicates that the max-min of the LCH of a function φ is attained in the interior of the domain of definition of the LCH. Formula (5.2) enables us to derive yet another equivalent representation for functions ψ^c_Δ with $c \geqslant 2\lambda K$, and also to establish that the max–min of the LCH of φ is Lipschitz continuous.

Property 5. If the function $\varphi(\cdot)$: $\bar{D}(t) \Rightarrow R$ satisfies the conditions of Theorem 2, the functions $\bar{\psi}^c_\Delta(\cdot)$: $\bar{D}(t) \Rightarrow R$, where $c \geqslant 2\lambda K$, may be constructed using the formula

$$\psi^c_\Delta(x) = \max_{y \in O(x, K_0\Delta)} \max_{s \in \partial\,\text{co}\,\varphi(y)} \{\Delta h(t, x, s) + \langle s, x - y \rangle + \text{co}\,\varphi(y)\}, \quad x \in \bar{D}(t) \tag{5.3}$$

where

$$\partial \operatorname{co}\varphi(y) = \{s \in R^n:\ \operatorname{co}\varphi(y^*) - \operatorname{co}\varphi(y) \geqslant \langle s, y^* - y \rangle$$

for any y^* in $O(x, K\Delta)$ is the subdifferential of the LCH of φ defined at a point y, where $y \in O(x, K_0\Delta)$.

The derivation of formula (5.3) uses techniques from convex analysis, relying on a criterion for a convex function to have a minimum subject to convex constraints [19]; it also uses the definition of the family (1.5). Note that representation (5.3) may be established using formulae for small displacements (see, for example, [16]).

Property 6. Let the function φ: $\overline{D}(t + \Delta) \Rightarrow R$ satisfy a Lipschitz condition with constant λ_φ. Then, for any $c \geqslant 2\lambda_\varphi K$, the function ψ^c_Δ: $\overline{D}(t) \Rightarrow R$ satisfies a Lipschitz condition with constant $\lambda_\varphi(1 + \Delta\lambda_F(1 + 3K))$

$$|\psi^c_\Delta(x) - \psi^c_\Delta(y)| \leqslant \lambda_\varphi(1 + \Delta\lambda_F(1 + 3K)) \| x - y \|$$

$x \in \overline{D}(t), y \in \overline{D}(t)$, and λ_F is the constant from condition Â4.

Let us consider the functions generated by the step operators π^c_Δ, $c \geqslant 0$. Let $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ be a partition of the interval $[t_0, \vartheta]$, and let $\varphi(\cdot)$: $\overline{D}(\vartheta) \Rightarrow R$ satisfy a Lipschitz condition with constant λ_φ. Consider the functions corresponding to this partition

$$\psi^c(\cdot): \bigcup_{i=1}^{N} (t_i, \overline{D}(t_i)) \Rightarrow R, \quad c \geqslant 0 \tag{5.4}$$

according to the recurrence relations

$$\psi^c(\vartheta, x) = \sigma(x), \quad x \in \overline{D}(\vartheta)$$
$$\psi^c(t_i, x) = \inf\{\chi:\ (x, \chi) \in \pi^c_{\Delta_i}(\operatorname{epi}\psi^c(t_{i+1}, \cdot), \quad x \in \overline{D}(t_i)\}$$
$$(\Delta_i = t_{i+1} - t_i,\ i = N-1,\ N-2, \ldots, 0)$$

We also define the successive max–min operator of the LCH. To that end we put

$$G_B(t, \Delta, \varphi)(x) = \max_{s \in S_n} \min_{f \in F(t,x,s)} \operatorname{co}\varphi(x + \Delta f), \quad x \in \overline{D}(t)$$

and let $\Phi(X)$ denote the set of all functions considered on a set X.

Definition 9. Let $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ be a partition of the interval $[t_0, \vartheta]$ and φ an arbitrary function defined on $\overline{D}(\vartheta)$. The successive max–min operator (SMO) is the operator $G(\Gamma, \varphi)$: $\Phi(\overline{D}(\vartheta)) \Rightarrow \Phi(\cup_{i=1}^{N} (t_i, \overline{D}(t_i)), t_i \in \Gamma$ defined by the relations

$$G(t_N, \varphi)(x) = \varphi(x), \quad x \in \overline{D}(\vartheta), \quad G(t_i, \ldots t_N, \varphi)(x) =$$
$$= G_B(t_i, \Delta_i, G_B(t_{i+1}, \Delta_{i+1}, (\ldots G_B(t_{N-1}, \Delta_{N-1}, \varphi)\ldots)))(x), \quad x \in \overline{D}(t_i), \quad i = N-1, \ldots, 0 \tag{5.5}$$

Properties 4–6 imply the following lemma.

Lemma. Let the function φ: $\overline{D}(\vartheta) \Rightarrow R$ satisfy a Lipschitz condition with constant λ_φ. Then, for any partition Γ of the interval $[t_0, \vartheta]$ and any values of the parameter $c \geqslant 2\lambda_\varphi \exp(\lambda_F(3K + 1)(\vartheta - t_0))$, the functions (5.4) corresponding to these parameter values are equal to one another and are identical with the function generated by the SMO

$$\psi^c(t_i, x) = G_B(t_i, \Delta_i, G_B(t_{i+1}, \Delta_{i+1}, (\ldots G_B(t_{N-1}, \Delta_{N-1}, \varphi)\ldots)))(x) \quad x \in \overline{D}(t_i), \quad t_i \in \Gamma$$

Theorems 1 and 2 and the lemma imply that a SMO approximates the solution of the Cauchy problem (1.1), (1.2).

Theorem 3. Assume that the function $\sigma(\cdot)$: $\overline{D}(\vartheta) \Rightarrow R$ satisfies a Lipschitz condition with constant λ_σ, the family of multivalued mappings $\{(t, x) \Rightarrow F(t, x, s) : s \in S_n\}$ satisfies conditions Â1–Â4, and $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ is a partition of the interval $[t_0, \vartheta]$ such that diam $\Gamma = \max_i \{|t_{i+1} - t_i| : i = 0, \ldots, N\} \to 0$ as $N \to +\infty$.

Then

$$|w(t_i, x) - G_B(t_i, \Delta_i, (\ldots G_B(t_{N-1}, \Delta_{N-1}, \sigma))\ldots)(x)| \to 0, \quad x \in \overline{D}(t_i), \quad t_i \in \Gamma$$

as diam $\Gamma \to 0$.

Remark 1. Similarly, when solving Problem 2, one constructs a successive min–max operator, $G^*(\Gamma, \varphi): \Phi(\overline{D}(\vartheta)) \Rightarrow \Phi(\cup_{i=1}^{N} (t_i, \overline{D}(t_i)), t_i \in \Gamma$, defined by

$$G^*(t_N, \varphi)(x) = \varphi(x), \quad x \in \overline{D}(\vartheta)$$
$$G(t_i, \ldots, t_N, \varphi)(x) = G_H(t_i, \Delta_i, G_H(t_{i+1}, \Delta_{i+1}, (\ldots G_H(t_{N-1}, \Delta_{N-1}, \varphi)\ldots))(x)$$
$$x \in \overline{D}(t_i), \quad i = N-1, \ldots, 0$$

where

$$G_H(t, \Delta, \varphi)(x) = \min_{s \in S_n} \max_{f \in F_*(t,x,s)} \operatorname{conc}\varphi(x + \Delta f), \quad x \in \overline{D}(t)$$

The family of multivalued mappings $\{(t, x) \Rightarrow F_*(t, x, s): s \in S_n\}$, where $F_*(t, x, s) = \{f \in F: \langle s, f \rangle \leq h(t, x, s)\}$, satisfies conditions analogous to Â1–Â4. The locally concave hull is defined by conc $\varphi(\cdot) = -\mathrm{co}(-\varphi(\cdot))$.

Remark 2. With additional assumptions (see [17, 18]), it can be shown that the operators G and G^* converge at a rate proportional to the square root of the diameter diam Γ of Γ.

6. THE COMPUTATION SCHEME. EXAMPLES

The basis of the computation scheme, implemented for the case of two space dimensions, consist of an operator $\overline{D}$ which is a finite-difference analogue of the operator G, considered over a uniform partition $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ of an interval $[t_0, \vartheta]$ of mesh-size Δ. The sets $\overline{D}(t_i)$, $t_i \in \Gamma$, are considered with rectangular meshes $\overline{D}(t_i, \alpha, \beta)$ of mesh-size $\alpha > 0$ in the variable x_1 and mesh-size β in the variable x_2

$$\overline{D}(t_i, \alpha, \beta) = \{x(k, d) \in \overline{D}(t_i) \subset R^2: \ x_1(k, d) = \hat{x}_1 + \alpha k$$
$$k = \pm 1, \pm 2, \ldots; \ x_2(k, d) = \hat{x}_2 + \beta d, \ d = \pm 1, \pm 2, \ldots\}$$

The operator

$$\overline{G}(\Gamma, \varphi): \ \Phi(\overline{D}(\vartheta, \alpha, \beta)) \Rightarrow \Phi\left(\bigcup_{i=1}^{N} (t_i, \overline{D}(t_i, \alpha, \beta))\right), \quad t_i \in \Gamma$$

is defined by

$$\overline{G}(t_N, \varphi)(x) = \varphi(x), \quad x \in \overline{D}(\vartheta, \alpha, \beta)$$
$$\overline{G}(t_i, \ldots, t_N, \varphi)(x) = \overline{G}_B(t_i, \Delta, \overline{G}_B(t_{i+1}, \Delta, (\ldots \overline{G}_B(t_{N-1}, \Delta, \varphi))\ldots))(x)$$
$$\overline{G}_B(t_i, \Delta, \varphi)(x(k, d)) = \max_{y \in O(x(k,d), K_0\Delta)} \max_{s \in \partial \mathrm{co}\varphi(y)} \{\Delta h(t_i, x(k, d), s) + \langle s, x(k, d) - y \rangle +$$
$$+ \mathrm{co}\varphi(y)\}, \quad x(k, d) \in \overline{D}(t_i, \alpha, \beta), \quad i = N-1, \ldots, 0$$

where $\overline{G}_B$ is a piecewise-linear approximation of G_B. In the event that the Hamiltonian is a piecewise-linear function and the function φ is defined on the mesh, the value of the operator $\overline{G}_B$ is computed as a successive maximum

$$\overline{G}_B(t_i,\Delta,\varphi)(x(k,d)) = \max_{x(j,m)} \max_{p} \max_{s} \{\Delta h_p(t_i,x(k,d),s) + \langle s, x(k,d)-x(j,m)\rangle + \mathrm{co}\varphi(x(j,m))\}$$

The maximization is carried out successively over the points $x(j, m) \in O(x(k, d), \Delta K_0$; over the numbers p defining the "pasting" function h_p $(t_i, x(k, d), s)$ = $(t_i, x(k, d), s)$, when s lies in the linearity cone $L_p(t_i, x(k, d))$ of the Hamiltonian with respect to s; and over the subgradient s of the linearity sets $L_p(t_i, x(k, d)x(j, m)) = L_p(t_i, x(k, d)) \cap \partial\mathrm{co}\ \varphi(x(j, m))$ of the operator G^-_B with respect to the variable s.

In numerical simulation, the construction of the convex hull of a function defined only at the nodes of a rectangular mesh presents the greatest difficulty. Here we appeal to the "gift-wrapping" algorithm developed for this case. To implement this algorithm one constructs a partition of the set of points of the graph of a function, when the latter is defined by tabular values, into disjoint subsets which can be convexified by elementary means (such as the method known as "Graham's scan"). The main part of the algorithm is a procedure for constructing the convexification of the union of two disjoint convex hulls. The procedure searches for a supporting edge and then implements one run of "gift-wrapping"—the successive construction of the facets of the convexification of the two convex hulls. The structure of the representation of the convex hull is a list of edges with dual bonds, which permits a fairly simple construction of the subdifferential of the convex hull. The intersections of the subdifferentials with the linearity cones of the Hamiltonian are constructed by solving a system of linear inequalities.

Example 1. (Test example [13].) Consider the Cauchy problem

$$\frac{\partial w}{\partial t} + x_2\frac{\partial w}{\partial x_1} + \left|\frac{\partial w}{\partial x_1}\right| - \left|\frac{\partial w}{\partial x_2}\right| = 0$$

$$w(2,x) = \max\{|x_1|, |x_2|\}, \quad t \in [0,2]$$

The numerical modelling is carried out at the point $\hat{x}$ = (0, 0) at a time t = 0.58578. The point $\hat{x}$ lies on the intersection of singular curves of the function $w(t, \cdot)$. We know that $w(t, \hat{x})$ = 0.58578. The parameters of the computation scheme and the corresponding approximate values w_a of the solution at the point $(t, \hat{x})$ are listed below

1) $\Delta = 0.28, \quad \alpha = 0.29, \quad \beta = 0.14, \quad w_a = 0.68$

2) $\Delta = 0.14, \quad \alpha = 0.29, \quad \beta = 0.14, \quad w_a = 0.66$

3) $\Delta = 0.09, \quad \alpha = 0.24, \quad \beta = 0.09, \quad w_a = 0.65$

Example 2. Consider a mathematical model of the control of the motion of a pendulum in a viscous medium

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 - \upsilon x_2 + v$$

where $t \in [0, 0.075]$, the "viscosity" is $v \in [0, 1]$, the control is such that $u \in [-1, 1]$, and the payoff function is $\sigma(x) = (x_1^2 + x_2^2)^{1/2}$.

No analytic solution of this problem is known. The results may be compared with those of a computation based on a different technique [14], which works by approximating level sets of the solution. Here we list the points $(t, \hat{x})$ at which the computation was performed, the parameters of the computation scheme and the corresponding approximate values w_a of the solution, as well as the approximate values $\overline{w}$ obtained by constructing level sets of the solution

1. the point $(t, \hat{x})$ = (0, 0.856, 0.755), for which $\overline{w}$ = 1

$\Delta = 0.15 \quad \alpha = 0.2, \quad \beta = 0.3, \quad w_a = 1.05$

$\Delta = 0.075, \quad \alpha = 0.15, \quad \beta = 0.15, \quad w_a = 1.01$

2. the point $(t, \hat{x})$ = (0, –1.05586, –0.36266), for which $\overline{w}$ = 1

$\Delta = 0.15, \quad \alpha = 0.2, \quad \beta = 0.3, \quad w_a = 1.04$

$\Delta = 0.075, \quad \alpha = 0.1, \quad \beta = 0.15, \quad w_a = 1.01$

Example 3. Consider the Cauchy problem

$$\frac{\partial w}{\partial t}+\sin x_2\frac{\partial w}{\partial x_1}-\exp(-x_1^2-x_2^2)\min\left\{0,\frac{\partial w}{\partial x_2}\right\}-\left|\frac{\partial w}{\partial x_2}\right|=0$$

$$w(0.5,x)=(x_1^2+x_2^2+0.81)^2-3.24x_1^2-1,\quad t\in[0;\,0.5]$$

The exact solution is unknown. As in Example 2, we list the points $(t,\hat{x})$ at which the computation is carried out, the parameters of the computation scheme and the corresponding values w_a of the solution, as well as the approximate values $\bar{w}$ obtained by constructing level sets of the solution

1. the point $(t,\hat{x})=(0,-0.551,-0.859)$, for which $\bar{w}=0$

$$\Delta=0.05,\quad \alpha=0.05,\quad \beta=0.1,\quad w_a=0.09$$
$$\Delta=0.025,\quad \alpha=0.025,\quad \beta=0.1,\quad w_a=0.04$$
$$\Delta=0.02,\quad \alpha=0.02,\quad \beta=0.04,\quad w_a=0.03$$

2. the point $(t,\hat{x})=(0,1.465,-0.082)$, for which $\bar{w}=0$

$$\Delta=0.05,\quad \alpha=0.05,\quad \beta=0.1,\quad w_a=0.14$$
$$\Delta=0.025,\quad \alpha=0.025,\quad \beta=0.05,\quad w_a=0.09$$
$$\Delta=0.02,\quad \alpha=0.02,\quad \beta=0.04,\quad w_a=0$$

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